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ONE REPRESENTATION OF THE CONDITIONS OF THE COMPATIBILITY OF DEFORMATIONS*

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It is shown that the conditions of the compatibility of deformations can be represented in the form of three equations in the region occupied by the deformed body, and three boundary conditions on its surface. A combination of the requirements of the conditions of equilibrium and campatibility leads to a unique formulation of the problem in terms of the stresses for the deformed body in the form of a system of six equations for the six unknown components of the stress tensor, and of a set of boundary conditions corresponding to the ninth order of the system of equations.

The classical formulation of the problem in terms of the stresses for a deformed rigid body leads to the need to solve a system containing its three equations of equilibrium and six compatability equations for six unknown components of the stress tensor. It can therefore be expected that some of the demands imposed by the formulation of the problem may be redundant. After all, such reasoning has been used systematically in similar situations in the scientific literature when formulating new problems, and was found to be effective.

1. We shall consider an elastic body occupying a three-dimensional region V, bounded by the surface S. We introduce in the region a Cartesian system of coordinates x_i with basis vector $\mathbf{e}_i$, so that the vector $\mathbf{n}$ normal to the surface S has components n_i. We shall denote differentiation with respect to the x_i coordinate by the index following the comma. We assume that the volume forces f_i and surface forces F_i are given. The mechanical properties of the material of the body in question will be described, generally speaking, by the following non-linear defining relations:

$$\varepsilon_{ij} = \varepsilon_{ij}(\sigma_{kl}) \tag{1.1}$$

connecting the deformation tensor ε_{ij} and stress tensor σ_{kl}.

The classical formulation of the boundary value problem of the mechanics of a deformable rigid body in terms of the stresses has certain specific features which merit attention. It is insufficient to satisfy three equations of equilibrium

$$\sigma_{ij,j} + f_i = 0, \quad \mathbf{x} \in V \tag{1.2}$$

with static boundary conditions

$$\sigma_{ij} n_j = F_i, \quad \mathbf{x} \in S \tag{1.3}$$

in order to determine uniquely the six components of the stress tensor σ_{ij}. Since the stresses are connected with the deformations by means of the defining Eqs.(1.1), the missing relations can be obtained from the natural geometrical Saint Venant conditions of compatibility for the components $\varepsilon_{ij}(\mathbf{x})$ of the deformation tensor. These can be written in the form /1/

$$g_{ij}(\mathbf{x}) \equiv \varepsilon_{ij,kk} + \varepsilon_{kk,ij} - \varepsilon_{ik,kj} - \varepsilon_{jk,ki}, \quad \mathbf{x} \in V \tag{1.4}$$

Although now we have more relations (1.4) than is necessary to formulate a definite system of equations for six functions σ_{ij} (or ε_{ij}) and the system (1.1) becomes overdefined

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in the sense used in /2/, the problem has not been discussed in the scientific literature. Sometimes straightforward assertions appear, as if it "could be confirmed" that all six equations of (1.4) were independent /1/. But, since the solution of the system (1.2)-(1.4) in question exists, on the one hand, and is unique on the other hand, we can expect that a certain three combinations of relations (1.4) can be discarded, even though we do not know which ones. Such an opinion was expressed e.g. by Donnell /1/ and Christensen /4/. We should remember, however, that discarding three compatibility relations (or their combinations) does not make the situation normal, since the remaining equations would form a ninth-order system, while the three boundary conditions (1.3) for the whole surface are characteristic of sixth-order systems.

In this connection we must not forget to mention the papers /5-7/ where * (*Vlasov B.F. Use of the two-sided energy approximation method in the statics of elastic construction elements. Candidate Dissertation, Moscow, 1974.) the authors give the relations of compatibility with three equations of continuity in integrodifferential form, such as, for example,

$$\begin{aligned}\varepsilon_{12} &= \int \varepsilon_{11,2}\, dx_1 + \int \varepsilon_{22,1}\, dx_2 + f_{1,2}(x_2, x_3) + f_{2,1}(x_1, x_3) \\ \varepsilon_{23} &= \int \varepsilon_{22,3}\, dx_2 + \int \varepsilon_{33,2}\, dx_3 + f_{2,3}(x_1, x_3) + f_{3,2}(x_1, x_2) \\ \varepsilon_{13} &= \int \varepsilon_{33,1}\, dx_3 + \int \varepsilon_{11,3}\, dx_1 + f_{3,1}(x_1, x_2) + f_{1,3}(x_2, x_3),\end{aligned} \tag{1.5}$$

where the indefinite integrals denote some particular primitive functions of the integrands, while $f_1(x_2, x_3)$, $f_2(x_1, x_3)$ and $f_3(x_1, x_2)$ are arbitrary functions of two corresponding variables. We must, however, note that while the incompatibility of the deformations $\varepsilon_{ij}(\mathbf{x})$ leads to explicit failure to satisfy the Saint Venant relations, in the case of relations (1.5) it leads to the non-existence of the functions f_i for which Eqs.(1.5) would hold. Thus the use of relations (1.5) in checking the compatibility of $\varepsilon_{ij}(\mathbf{x})$ reduces to solving the problem of the existence or non-existence of the solution of the system (1.5) for f_i $(i = 1, 2, 3)$, i.e. to solving a problem of the same type as that arising in the study of the initial problem, namely whether three functions u_i representing the soltuion of the Cauchy relations $u_{i,j} + u_{j,i} = 2\varepsilon_{ij}$ do exist.

In this sense, it would be more correct not to call relations (1.5) the equations of continuity (compatibility), unlike the Saint Venant relations where satisfying (or not satisfying) the latter explicitly is the necessary and sufficient condition for the compatibility (or incompatibility) of the deformations ε_{ij}.

2. We will show that the six compatibility relations (1.4) are not mutually independent. Indeed, from the definition of g_{ij} (1.4) we have

$$g_{ij,j} = \varepsilon_{jj,kki} - \varepsilon_{jk,jki}, \quad g_{jj} = 2(\varepsilon_{jj,kk} - \varepsilon_{jk,jk})$$

Therefore the six conditions of compatibility (1.4) are connected by three relations

$$2g_{ij,j} = g_{jj,i} \tag{2.1}$$

3. We shall now show that instead of (1.4) we can use the following three conditions within the region as the necessary and sufficient conditions of compatibility of the deformations $\varepsilon_{ij}(\mathbf{x})$:

$$g_{12}(\mathbf{x}) = 0, \quad g_{23}(\mathbf{x}) = 0, \quad g_{31}(\mathbf{x}) = 0, \quad \mathbf{x} \in V \tag{3.1}$$

and three conditions on the boundary (no summation over i)

$$2g_{ii}(\mathbf{x}) - g_{jj}(\mathbf{x}) = 0, \quad \mathbf{x} \in S_{i-}; \quad i = 1, 2, 3 \tag{3.2}$$

where S_{i-} is the part of the boundary of the body for which the inequality $\mathbf{n}\cdot\mathbf{e}_i \leqslant 0$ holds.

For every internal point $\mathbf{x} \in V$ its projections $\mathbf{x}_{(i)} = x_{(i)k}\mathbf{e}_k \in S_{i-}$ in the direction $\mathbf{e}_i$ on the part of the boundary S_{i-} exist such that all points $x_{(i)}' = x_{(i)k}'\mathbf{e}_k$ on the segments connecting $\mathbf{x}_{(i)}$ with $\mathbf{x}$, described by the expression

$$\mathbf{x}_{(i)}' = \mathbf{x} + (x_{(i)i}' - x_i)\,\mathbf{e}_i, \quad x_{(i)i} \leqslant x_{(i)i}' \leqslant x_i$$

in which there is no summation over the repeated indices, are within the body, i.e. $\mathbf{x}_{(i)}' \in V$.

The following relation holds for the pair of points $\mathbf{x}$ and $\mathbf{x}_1$ by virtue of (2.1) $i = 1$:

$$2g_{11}(\mathbf{x}) - g_{jj}(\mathbf{x}) = 2g_{11}(\mathbf{x}_{(1)}) - g_{jj}(\mathbf{x}_{(1)}) - 2\int_{x_{(1)1}}^{x_1} [g_{12,2}(\mathbf{x}_{(1)}') + g_{13,3}(\mathbf{x}_{(1)}')]\, dx_{(1)1}'$$

Analogous equations hold for the pairs of points $\mathbf{x}, \mathbf{x}_2$ and $\mathbf{x}, \mathbf{x}_3$. Therefore Eqs.(3.2) hold for $x \in V$, provided that conditions (3.1) and (3.2) are satisfied. From this it follows that $g_{jj}(\mathbf{x}) = 0$, $g_{11}(\mathbf{x}) = 0$, $g_{22}(\mathbf{x}) = 0$, $g_{33}(\mathbf{x}) = 0$, $\mathbf{x} \in V$ also holds, so that relations (1.4) hold for all points of the body $\mathbf{x} \in V$. The converse, that (1.4) hold if (3.1), (3.2) hold, is trivial. Thus we find that in addition to the usual form of representing the necessary and sufficient conditions of compatibility of the deformations $\varepsilon_{ij}(\mathbf{x})$ in the form of the relations (1.4), we can write them in the form (3.1), (3.2).

It is clear that all assertions made will remain true if we replace in (3.2) the surfaces S_{i-} by the parts S_{i+} of the surface S of the body for which the inequalities $\mathbf{n}\cdot\mathbf{e}_i \geqslant 0$ hold.

4. Assuming that the deformations ε_{ij} are expressed in terms of the stresses σ_{ij} in accordance with the defining relations for the material of the body in question (1.1), we arrive at the unique formulation of the problem in terms of the stresses, in the form of equations (1.2), (3.1) and boundary conditions (1.3), (3.2). In this case we have, as we always desired, six equations for the six unknown functions σ_{ij}. The sum of the doubled number of boundary conditions (1.3) specified over the whole surface S of the body, and the number of boundary conditions (3.2) specified on the parts S_{i-} or S_{i+} of the surface S corresponds, as would be expected, to the ninth order of the system (1.2), (3.1). Some unease about the fact that the boundary of the boundary conditions (3.2) are specified on a part and not on the whole of the surface S, can be dispelled by pointing out that the order of the system is odd and that this is characteristic of such situations.

It is interesting, from the same point of view, to compare the problem formulated in terms of the stresses with that of /8/. If we assume that the deformations are expressed in terms of the stresses with the help of the defining Eqs.(1.1), and then introduce a symmetric constant tensor ξ_{ij} and some symmetric tensor-operator $Q_{ij}(S_{kl})$ of tensor-argument $S_{kl} = \sigma_{kn,nl} + \sigma_{ln,nk} + f_{k,l} + f_{l,k}$, and assume at the same time that $\xi_{nn} \neq 2$ and the relation connecting Q_{ij} with S_{kl} satisfies the same properties as are usually assumed satisfied for the defining Eqs. (1.1), then the stresses σ_{kl}, with the volume forces f_i acting within the body V and forces F_i acting at its surface S, represent the solution of the twelfth-order system of six Eqs.(8)

$$\varepsilon_{ij,nn} + \varepsilon_{nn,ij} - \varepsilon_{in,nj} - \varepsilon_{jn,ni} + \xi_{ij}(\varepsilon_{mn,mn} - \varepsilon_{mm,nn}) + Q_{ij} + (\xi_{ij} - \delta_{ij}) Q_{nn} = 0, \quad \mathbf{x} \in V$$

with boundary conditions (1.3) and

$$\sigma_{ij,j} + f_i = 0, \quad \mathbf{x} \in S$$

The advantage of such a formulation as compared with the classical formulation consists of the fact that the number of equations matches the number of unknown functions, and the doubled number of boundary conditions specified on the whole surface S of the body corresponds to the order of the system. At the same time we can say that the formulation (1.2), (3.1), (1.3), (3.2) of the problem is of real interest, due to the fact that the resulting system of equations is of lower order, that the formulation arises naturally as a simple combination of the demands that the conditions of equilibrium and compatibility hold, and that it is no longer necessary to introduce, during the formulation stage, the quantitative characteristics of the tensor ξ_{ij} and tensor-operator $Q_{ij}(S_{kl})$, which are irrelevant to the problem in hand.

It appears that since the order of differentiation in the conditions of compatibility (3.1), (3.2) is higher than that in conditions of equilibrium (1.2), (1.3), it is more convenient from the practical point of view to formulate the problem in terms of the deformations (1.2), (3.1), (1.3), (3.2), assuming that the stresses in (1.2) and (1.3) expressed in terms of the deformations in accordance with (1.1) have a solution in terms of the stresses

$$\sigma_{ij} = \sigma_{ij}(\varepsilon_{kl}) \tag{4.1}$$

except in the case of a homogeneous, isotropic elastic body.

Since the classical representations of the conditions of compatibility (1.4) and the representation (3.1), (3.2) obtained above are completely equivalent, it follows that a solution of problem (1.2), (3.1), (1.3), (3.2) exists and is unique when the conditions regarding the loads f_i and F_i and the properties of the defining relations (1.1) are the same as in the classical formulation of the problem.

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